SOLVABLE MODELS FOR THE SCHRÖDINGER OPERATORS WITH δ'-LIKE POTENTIALS

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ABSTRACT. We turn back to the well known problem of interpretation of the Schrödinger operator with the pseudopotential $\alpha\delta'(x)$, where $\delta(x)$ is the Dirac function and $\alpha \in \mathbb{R}$. We show that the problem in its conventional formulation contains hidden parameters and the choice of the proper selfadjoint operator is ambiguously determined. We study the asymptotic behavior of spectra and eigenvectors of the Hamiltonians with increasing smooth potentials perturbed by short-range potentials $\alpha\varepsilon^{-2}\Psi(\varepsilon^{-1}x)$. Appropriate solvable models are constructed and the corresponding approximation theorems are proved. We introduce the concepts of the resonance set Σ_Ψ and the coupling function $\theta_\Psi : \Sigma_\Psi \to \mathbb{R}$, which are spectral characteristics of the potential Ψ. The selfadjoint operators in the solvable models are determined by means of the resonance set and the coupling function.

INTRODUCTION

In this paper we investigate the one-dimensional Schrödinger operators with the coefficients having singular support on a disjoint set of points. Such operators are named *operators with singular point interactions* and are closely related to exactly solvable problems in quantum mechanics, atomic physics and acoustics. Such point interaction models are *solvable* in the sense that the resolvents of corresponding operators can be given explicitly. As a consequence the spectrum and scattering quantities can also be determined explicitly.

The main question here is how to interpret the differential operators with coefficients equal to the generalized functions since space of distributions $\mathcal{D}'(\mathbb{R}^n)$ is not an algebra with respect to the "pointwise" multiplication. Generally the problem of choosing a single Hamiltonian corresponding to a particular singular point interaction is complicated enough. The theory of one-dimensional Schrödinger operators with singular potentials has been summarized in research monographs [1, 3].

According to the postulates of quantum mechanics every physically measurable quantity is specified by a *selfadjoint* operator in a Hilbert space.

Starting with the pioneering work of F. Berezin and L. Faddeev [8], the singular point interactions have been studied with the help of the selfadjoint extensions theory for symmetric operators. As a rule, the symmetric operators in physical models are generated by differential operators. This leads to the question how to describe all selfadjoint extensions in terms of the boundary value conditions. In the case of ordinary differential operators the problem has been solved by M. Krein [7, 15].

For partial differential operators the problem becomes more complicated by the infinite deficiency indices of symmetric operators. In that case new approach to the theory of selfadjoint extensions is based on the concept of boundary triples [23, 24] and the corresponding Weyl functions. In [16, 17] the concept of Weyl function was generalized to an arbitrary symmetric operator with infinite deficiency indices. It is well known that the Weyl function can be used to investigate the spectrum of a selfadjoint extension and the scattering matrix [9]. It is worth to note that the concept of boundary triples is useful in the investigation of the point interaction models [22, 36, 37, 38].

The singular finite rank perturbations have been studied in papers [2, 39, 40, 46] and monograph [41]. The Schrödinger operators with different singular potentials have been investigated in [31, 45, 47, 53, 54, 55].

In some instances the set of all selfadjoint extensions of a symmetric operators is rich enough. Therefore the harder question comes: how to choose an extension that is best suited to our physical model. For some models the proper operator can not be chosen within the selfadjoint extensions theory, because the models contain hidden parameters. If we replace the singular potential with a sequence of short-range smooth potentials, then the operator obtained in the zero-range limit, as often happens, can depend on the type of regularization, that is to say, the operator is governed by the profile of squeezed potentials. This profile is a hidden parameter and plays a crucial role in the choice of a selfadjoint extension corresponding to the physical model under consideration.

Differential operators with singular coefficients can be studied within the framework of the contemporary theories such as the Colombeau algebras [12] or the Egorov generalized function algebra [28]. These theories allow us not only to multiply the generalized functions, but they also contain rich sets of "δ-functions with fixed profiles" as different elements of the algebra. This approach to study of solvable models has been used in [5, 6].

1. Problem Statement and Main Results

1.1. **Problem statement.** In this work we turn back to the well-known problem how to define the one-dimensional Schrödinger operator with the δ'-potential. New attempt will be made to answer this question.

Let us consider the formal Hamiltonian

$$H_\alpha = -\frac{d^2}{dx^2} + U(x) + \alpha\delta'(x), \qquad x \in \mathbb{R},$$

where U is a real smooth function, δ' is the first derivative of the Dirac delta function, and α is a real constant. If product $\delta'(x)y(x)$ is treated as $y(0)\delta'(x) - y'(0)\delta(x)$, then for $\alpha \neq 0$ there exist no solutions in $\mathcal{D}'(\mathbb{R})$ to equation $H_\alpha y = \lambda y$, except for a trivial one. In order for operator H_α to be assigned a meaning, firstly we must describe all selfadjoint extensions in $L_2(\mathbb{R})$ of the symmetric operator

$$L = -\frac{d^2}{dx^2} + U(x), \qquad \mathcal{D}(L) = \{f \in C_0^\infty(\mathbb{R}) \colon f(0) = f'(0) = 0\}.$$

Next, one of the extensions must be prescribed for the Schrödinger operators with potential $U(x) + \alpha\delta'(x)$. At present, in both physics- and mathematics-oriented literature there is no a consensus regarding the proper choice of such operator, since L has a rich set of selfadjoint extensions (see Sec. 4 for the historical remarks).

Let us consider more realistic model, namely the family of Hamiltonians with smooth short-range potentials, which approximate the singular potential $\alpha\delta'(x)$. Let $\mathcal{H}_\varepsilon(\alpha, \Psi)$ be the closure in $L_2(\mathbb{R})$ of the essentially selfadjoint operator [10, p. 50]

$$H_\varepsilon(\alpha, \Psi) = -\frac{d^2}{dx^2} + U(x) + \frac{\alpha}{\varepsilon^2}\Psi(\varepsilon^{-1}x), \quad \mathcal{D}(H_\varepsilon(\alpha, \Psi)) = C_0^\infty(\mathbb{R}).$$

Here ε is a small positive parameter. We call $\Psi \in C_0^\infty(\mathbb{R})$ the *profile* of a local perturbation, and α the *coupling constant*. It is clear that for some profiles sequence $\varepsilon^{-2}\Psi(\varepsilon^{-1}x)$ converges to $\delta'(x)$ in $\mathcal{D}'(\mathbb{R})$. We will denote by $\mathcal{P}$ the set of real functions $\Psi \in C_0^\infty(\mathbb{R})$ such that $\operatorname{supp}\Psi = [-1, 1]$. Let $\mathcal{E}(L)$ denote the set of all selfadjoint extensions of L. Suppose also that potential U increases as $|x| \to \infty$. It follows that the spectrum of $\mathcal{H}_\varepsilon(\alpha, \Psi)$ is discrete.

Our main goal is to construct the map $\mathbb{R} \times \mathcal{P} \longrightarrow \mathcal{E}(L)$ that assigns a selfadjoint extension $\mathcal{H}(\alpha, \Psi)$ of operator L to each pair (α, Ψ). The choice of an operator is determined by the proximity of the energy levels and the pure states for the Hamiltonians with smooth and singular potentials respectively. We explore the asymptotic behavior of eigenvalues and eigenfunctions as $\varepsilon \to 0$. The asymptotics provides the limit operator $\mathcal{H}(\alpha, \Psi)$.

1.2. **Structure of Paper.** The paper is organized as follows. In Sec. 2, we derive qualitative properties of the discrete spectrum of perturbed operators $\mathcal{H}_\varepsilon(\alpha, \Psi)$. We show that all eigenvalues are continuous functions on ε that are bounded from above. Generally speaking, the spectrum of this family is not bounded from below. For some profiles Ψ and $\alpha \neq 0$ there exists a finite number of eigenvalues which go to $-\infty$ as $\varepsilon \to 0$. By the *bounded spectrum*

of $\mathcal{H}_\varepsilon(\alpha, \Psi)$ we mean the set of such eigenvalues that remain bounded as $\varepsilon \to 0$.

The leading terms of asymptotic expansions for the eigenvalues of the bounded spectrum and the limit operators $\mathcal{H}(\alpha, \Psi)$ are formally constructed in Sec. 3. We introduce two spectral characteristics of profile Ψ, namely the *resonance set* Σ_Ψ that is the spectrum of the Sturm–Liouville problem $-w'' + \alpha \Psi w = 0$, $w'(-1) = 0$, $w'(1) = 0$ on interval $(-1, 1)$ with respect to spectral parameter α, and the *coupling function* $\theta_\Psi \colon \Sigma_\Psi \to \mathbb{R}$. We set $\theta_\Psi(\alpha) = w_\alpha(1)\big(w_\alpha(-1)\big)^{-1}$, where w_α is an eigenfunction corresponding to eigenvalue $\alpha \in \Sigma_\Psi$. In the case, when the coupling constant doesn't belong to the resonant set, $\mathcal{H}(\alpha, \Psi)$ is just the direct sum of the Schrödinger operators with potential U on half-axes subject to the Dirichlet boundary condition at the origin. In the resonant case, when $\alpha \in \Sigma_\Psi$, operator $\mathcal{H}(\alpha, \Psi)$ is the closure in $L_2(\mathbb{R})$ of the essentially selfadjoint operator $H(\alpha, \Psi) = -\frac{d^2}{dx^2} + U$ with domain $\mathcal{D}(H(\alpha, \Psi)) = \{f \in C_0^\infty(\mathbb{R}) \colon f(+0) = \theta_\Psi(\alpha)f(-0), \ \theta_\Psi(\alpha)f'(+0) = f'(-0)\}$.

In Sec. 4 we provide a detail discussion of the previous results dealing with the δ'-interactions and the δ'-potentials. We give two examples of exactly solvable models with piecewise constant δ'-like potentials for which the resonance set and the coupling function are explicitly computed. We solve the scattering problem for this δ'-like potential. The limit value of the transmission coefficient can be also explicitly computed in terms of coupling function θ_Ψ.

The remainders of asymptotics for the eigenvalues and eigenfunctions of $\mathcal{H}_\varepsilon(\alpha, \Psi)$ are constructed in Sec. 5, because we are in need of more precise asymptotics, in order to proof the approximation theorems.

Section 6 contains the main results of this paper. We prove in this section that the eigenvalues of $\mathcal{H}_\varepsilon(\alpha, \Psi)$ that belong to the bounded spectrum converge to eigenvalues of $\mathcal{H}(\alpha, \Psi)$. The convergence in $L_2(\mathbb{R})$ of the corresponding eigenfunctions is also established.

2. Spectrum of $\mathcal{H}_\varepsilon(\alpha, \Psi)$ and Auxiliary Results

Firstly we describe the set $\mathcal{E}(L)$ of all selfadjoint extensions of minimal operator L. The adjoint operator $L^* = -\frac{d^2}{dx^2} + U(x)$ is defined on the domain

$$\mathcal{D}(L^*) = \{v \in W_2^2(\mathbb{R} \setminus 0) \colon -v'' + Uv \in L_2(\mathbb{R})\}.$$

Lemma 2.1. *Every element of $\mathcal{E}(L)$ coincides with operator L^*, restricted to the set of functions, satisfying the boundary conditions at the origin of one of the types*

$$(2.1) \qquad h_1^- v'(-0) = h_2^- v(-0), \qquad h_1^+ v'(+0) = h_2^+ v(+0)$$

with the parameters $(h_1^\pm, h_2^\pm)$ from the projective space $\mathbb{P}^1$;

$$(2.2) \qquad \begin{pmatrix} v(+0) \\ v'(+0) \end{pmatrix} = C \begin{pmatrix} v(-0) \\ v'(-0) \end{pmatrix}, \qquad C = e^{i\varphi} \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix},$$

where $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $c_{kl} \in \mathbb{R}$ and $c_{11}c_{22} - c_{12}c_{21} = 1$.

This lemma has been proved in [11, 50] for $U = 0$. Obviously, the smooth potential does not affect on the boundary conditions at the origin. Selfadjoint operators described by boundary conditions (2.1) will be named *separated extensions*. These operators are equal to the orthogonal sum of two selfadjoint operators defined on the half-axes. Selfadjoint operators of the second type will be named *connected extensions*, because these conditions connect the boundary values of the function on the left and right half-axes.

Let us write $\Psi_\varepsilon(x) = \varepsilon^{-2}\Psi(\varepsilon^{-1}x)$ and set $m_k(f) = \int_{\mathbb{R}} \xi^k f(\xi)\, d\xi$.

Lemma 2.2. *Assume $\Psi \in \mathcal{P}$ and c is a nonzero constant. Sequence Ψ_ε converges to $c\delta'(x)$ as $\varepsilon \to 0$ in the sense of distributions if and only if*

$$(2.3) \qquad\qquad m_0(\Psi) = 0, \quad m_1(\Psi) \neq 0.$$

In addition, $c = -m_1(\Psi)$.

*Proof.* Given $\varphi \in C_0^\infty(\mathbb{R})$, we have

$$\langle \Psi_\varepsilon, \varphi \rangle = \varepsilon^{-2} \int\limits_{-\varepsilon}^{\varepsilon} \Psi(\varepsilon^{-1}x)\varphi(x)\, dx = \varepsilon^{-1} \int\limits_{-1}^{1} \Psi(\xi)\varphi(\varepsilon\xi)\, d\xi$$

$$= \varepsilon^{-1} \int\limits_{-1}^{1} \Psi(\xi)\big(\varphi(0) + \varepsilon\varphi'(0)\xi + O(\varepsilon^2)\big)\, d\xi$$

$$= \varepsilon^{-1}m_0(\Psi)\varphi(0) + m_1(\Psi)\varphi'(0) + O(\varepsilon)$$

as $\varepsilon \to 0$. Hence sequence $\langle \Psi_\varepsilon, \varphi \rangle$ converges for all $\varphi \in C_0^\infty(\mathbb{R})$ to a finite limit if and only if $m_0(\Psi) = 0$. Moreover if $m_1(\Psi) \neq 0$, then the limit is nontrivial and the limit functional is defined by $\Psi_0 = -m_1(\Psi)\delta'(x)$, since $\langle \Psi_0, \varphi \rangle = m_1(\Psi)\varphi'(0)$. $\qquad\qquad\square$

The first condition in (2.3) implies that function Ψ must change a sign on $[-1, 1]$. The second one involves an asymmetric property of this function: if $\Psi(\xi)$ satisfies (2.3), then $\Psi(\xi - a)$ cannot be an even function for any a. We introduce the set of δ'-*like potentials*

$$\mathcal{P}_0 = \{\Psi \in \mathcal{P} \colon m_0(\Psi) = 0,\ m_1(\Psi) = -1\}.$$

Recall that $U(x) \to +\infty$ as $|x| \to +\infty$. Then for all $\varepsilon > 0$ the spectrum of $\mathcal{H}_\varepsilon(\alpha, \Psi)$ is real, discrete and simple. Let $\{\lambda_k^\varepsilon(\alpha, \Psi)\}_{k=1}^\infty$ be the eigenvalues of $\mathcal{H}_\varepsilon(\alpha, \Psi)$ enumerated in increasing. Suppose that $\{y_k^\varepsilon(x; \alpha, \Psi)\}_{k=1}^\infty$ is the orthonormal in $L_2(\mathbb{R})$ system of eigenfunctions.
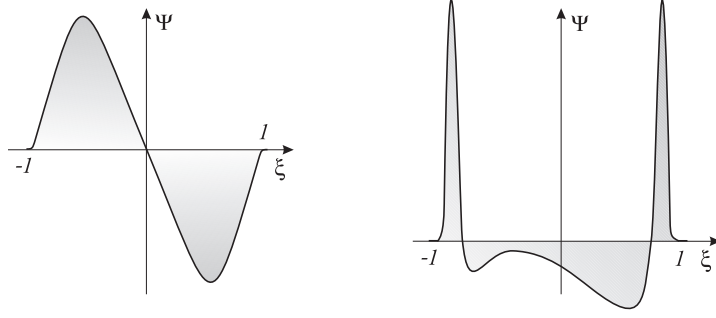
FIGURE 1. Plots of δ'-like profiles

Theorem 2.1. *For each pairs $(\alpha, \Psi) \in \mathbb{R} \times \mathcal{P}$ eigenvalues $\lambda_k^\varepsilon(\alpha, \Psi)$ are continuous functions with respect to $\varepsilon \in (0,1)$. Moreover, the eigenvalues are bounded from above as $\varepsilon \to 0$. In the case of the δ'-like potential, the spectrum of $\mathcal{H}_\varepsilon(\alpha, \Psi)$ is unbounded from below as $\varepsilon \to 0$: if $\alpha \neq 0$ and $\Psi \in \mathcal{P}_0$, then $\lambda_1^\varepsilon(\alpha, \Psi) \leq -\gamma \varepsilon^{-2}$ for some positive constant γ. There is only finite number $N(\alpha, \Psi)$ of eigenvalues that converge to $-\infty$ as $\varepsilon \to 0$.*

Proof. Fix α and Ψ. The quadratic form

$$a_\varepsilon[u] = \int_{\mathbb{R}} \left(|u'|^2 + (U + \alpha\Psi_\varepsilon)|u|^2 \right) dx, \qquad u \in C_0^\infty(\mathbb{R})$$

is a continuous function with respect to $\varepsilon \in (0,1)$. The minimax principle

$$\lambda_k^\varepsilon(\alpha, \Psi) = \inf_{E_k} \sup_{v \in E_k, \, \|v\|=1} a_\varepsilon[v]$$

yields the continuity of eigenvalues with respect to ε. Here E_k is a k-dimensional linear subspace of $C_0^\infty(\mathbb{R})$, and $\|\cdot\|$ denotes the norm in $L_2(\mathbb{R})$.

We choose subspace E_k^*, whose elements vanish in a neighborhood of the origin. Then, we have

(2.4) $$\lambda_k^\varepsilon(\alpha, \Psi) \leq \sup_{v \in E_k^*, \, \|v\|=1} a_\varepsilon[v].$$

For all sufficiently small ε the restriction of a_ε to the finite-dimensional space E_k^* does not depend on ε. This yields the boundedness of the eigenvalues from above.

From now on, Ψ is a δ'-like profile. It is well known [51, p. 338] that the Schrödinger operator $-\frac{d^2}{d\xi^2} + \alpha\Psi(\xi)$ on the line has a negative eigenvalue for all $\alpha \neq 0$ if and only if $m_0(\Psi) = 0$. Hence there exists function u, $\|u\| = 1$, such that

$$\mu = \int_{\mathbb{R}} \left(u'^2 + \alpha\Psi(\xi)u^2 \right) d\xi < 0.$$

Let $\zeta \in C_0^\infty(\mathbb{R})$ be the positive cutoff function such that $\operatorname{supp} \zeta = [-2, 2]$ and $\zeta(x) = 1$ for $x \in [-1, 1]$. For abbreviation, we write ζ_ε and ζ'_ε instead of $\zeta(\varepsilon\xi)$ and $\zeta'(\varepsilon\xi)$ respectively. Let us consider sequence $u_\varepsilon(x) = \varepsilon^{-1/2}\zeta(x)u(\varepsilon^{-1}x)$. Then

$$\|u_\varepsilon\|^2 = \frac{1}{\varepsilon}\int_{\mathbb{R}} \zeta^2(x)u^2(\tfrac{x}{\varepsilon})\,dx = \int_{\mathbb{R}} \zeta_\varepsilon^2(\xi)u^2(\xi)\,d\xi \to \int_{\mathbb{R}} u^2(\xi)\,d\xi = 1$$

as $\varepsilon \to 0$. Next, we conclude from the minimax principle that

$$\lambda_1(\varepsilon, \alpha) = \inf_{\substack{v \in C_0^\infty, \\ \|v\|=1}} a_\varepsilon[v] \le \frac{a_\varepsilon[u_\varepsilon]}{\|u_\varepsilon\|},$$

hence that

$$\varepsilon^2\|u_\varepsilon\|\,\lambda_1(\varepsilon, \alpha) \le \varepsilon^2 a_\varepsilon[u_\varepsilon]$$

$$= \varepsilon \int_{\mathbb{R}} \left(\left(\zeta(x)u(\varepsilon^{-1}x)\right)'^2 + (U + \alpha\Psi_\varepsilon)(\zeta(x)u(\varepsilon^{-1}x))^2\right)\,dx$$

$$= \int_{\mathbb{R}} \zeta_\varepsilon^2\left(u'^2 + \alpha\Psi u^2\right)\,d\xi + 2\varepsilon \int_{\mathbb{R}} \zeta_\varepsilon\zeta'_\varepsilon uu'\,d\xi$$

$$+ \varepsilon^2 \int_{\mathbb{R}} \zeta_\varepsilon'^2 u^2\,d\xi + \varepsilon \int_{-2}^{2} U(x)\zeta^2(x)u^2(\varepsilon^{-1}x)\,dx.$$

It is easily seen that the first integral converges to negative number μ as $\varepsilon \to 0$, and other terms go to zero. Thus for ε sufficiently small the estimate $\varepsilon^2\|u_\varepsilon\|\,\lambda_1(\varepsilon, \alpha) \le \mu/2$ holds, and so $\lambda_1(\varepsilon, \alpha) \le -\gamma\varepsilon^{-2}$ for some $\gamma > 0$.

Let $N_\varepsilon^-(\alpha, \Psi)$ be the number of negative eigenvalues of $\mathcal{H}_\varepsilon(\alpha, \Psi)$. Obviously $N(\alpha, \Psi) \le N_\varepsilon^-(\alpha, \Psi)$ for $\varepsilon > 0$ sufficiently small. In the case of the continuous potential the inequality

$$(2.5) \qquad N_\varepsilon^-(\alpha, \Psi) \le 1 + \int_{\mathbb{R}} |x|\,|U^-(x)|\,dx + |\alpha|\int_{\mathbb{R}} |x|\,|\Psi_\varepsilon^-(x)|\,dx$$

holds [10, p.97], where $f^-(x) = \min\{f(x), 0\}$ is the negative part of f. Since U increases as $x \to \infty$, U^- is a function with compact support, and consequently the first integral in (2.5) is defined. The support of Ψ_ε^- lies in $[-\varepsilon, \varepsilon]$, we thus deduce

$$\int_{\mathbb{R}} |x|\,|\Psi_\varepsilon^-(x)|\,dx = \varepsilon^{-2}\int_{-\varepsilon}^{\varepsilon} |x|\,|\Psi^-(\tfrac{x}{\varepsilon})|\,dx = \int_{-1}^{1} |\xi|\,|\Psi^-(\xi)|\,d\xi.$$

In view of (2.5) and the formula above $N(\alpha, \Psi)$ is uniformly bounded with respect to ε:

$$N(\alpha, \Psi) \leq N_\varepsilon^-(\alpha, \Psi) \leq c_1(U) + c_2(\Psi)|\alpha|$$

with constants $c_1(U)$, $c_2(\Psi)$ being positive. $\square$

Therefore the spectrum of $\mathcal{H}_\varepsilon(\alpha, \Psi)$ consists of two parts: $\{\lambda_k^\varepsilon(\alpha, \Psi)\}_{k=1}^N$ is the set of eigenvalues that tend to $-\infty$ as $\varepsilon \to 0$, and $\{\lambda_k^\varepsilon(\alpha, \Psi)\}_{k=N+1}^\infty$ is the set of bounded eigenvalues, which we call the bounded spectrum of $\mathcal{H}_\varepsilon(\alpha, \Psi)$. Notice that eigenfunctions $y_1^\varepsilon(x; \alpha, \Psi), \ldots, y_N^\varepsilon(x; \alpha, \Psi)$ have strongly oscillatory character as $\varepsilon \to 0$ in an neighborhood of the origin and vanish exponentially outside. They converge to zero in $L_2(\mathbb{R})$ weakly.

3. Asymptotics of Bounded Spectrum of $\mathcal{H}_\varepsilon(\alpha, \Psi)$: Leading Terms

Let us consider the eigenvalue problem

$$(3.1) \qquad -y_\varepsilon'' + \left(U(x) + \alpha\varepsilon^{-2}\Psi(\varepsilon^{-1}x)\right) y_\varepsilon = \lambda^\varepsilon y_\varepsilon, \qquad y_\varepsilon \in L_2(\mathbb{R}).$$

Fix eigenvalue $\lambda_k^\varepsilon(\alpha, \Psi)$ with number $k > N(\alpha, \Psi)$. We denote it briefly by λ^ε. Let y_ε be the corresponding eigenfunction. We postulate asymptotic expansions for the eigenvalue

$$(3.2) \qquad \lambda^\varepsilon \sim \lambda + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \cdots,$$

and two-scale expansions for the eigenfunction

$$(3.3) \quad y_\varepsilon(x) \sim v(x) + \varepsilon\, v_1(x) + \varepsilon^2 v_2(x) + \cdots \qquad \text{for } |x| > \varepsilon,$$

$$(3.4) \quad y_\varepsilon(x) \sim w(\varepsilon^{-1}x) + \varepsilon\, w_1(\varepsilon^{-1}x) + \varepsilon^2 w_2(\varepsilon^{-1}x) + \cdots \qquad \text{for } |x| \leq \varepsilon.$$

Here v, v_i are defined for $x \in \mathbb{R} \setminus \{0\}$ and belong to $L_2(\mathbb{R})$. Set $\xi = \varepsilon^{-1}x$. Functions w, w_i are defined for $\xi \in [-1, 1]$. We also assume that v is deferent from zero. Series (3.3), (3.4) satisfy the coupling conditions

$$(3.5) \qquad\qquad [y_\varepsilon]_{x=\pm\varepsilon} = 0, \quad [y_\varepsilon']_{x=\pm\varepsilon} = 0$$

at points $x = \pm\varepsilon$, where $[f]_{x=a} = f(a+0) - f(a-0)$ is the jump of f at point a. This type of combined asymptotic expansions has been used in particular in [18]-[20].

Upon substituting (3.2)-(3.4) into equation (3.1), we obtain

$$(3.6) \qquad -v'' + U(x)v = \lambda v, \qquad x \in \mathbb{R} \setminus \{0\},$$

$$(3.7) \qquad -v_1'' + U(x)v_1 = \lambda v_1 + \lambda_1 v, \qquad x \in \mathbb{R} \setminus \{0\},$$

and also

$$(3.8) \qquad -w'' + \alpha \Psi(\xi) w = 0, \qquad \xi \in (-1, 1),$$

$$(3.9) \qquad -w_1'' + \alpha \Psi(\xi) w_1 = 0, \qquad \xi \in (-1, 1),$$

$$(3.10) \qquad -w_2'' + \alpha \Psi(\xi) w_2 = \lambda w - U(0) w, \qquad \xi \in (-1, 1).$$

Next, substituting series (3.3), (3.4) into the coupling conditions yields

$$v(\pm\varepsilon) + \varepsilon\, v_1(\pm\varepsilon) + \cdots \sim w(\pm 1) + \varepsilon\, w_1(\pm 1) + \cdots\,,$$

$$v'(\pm\varepsilon) + \varepsilon\, v_1'(\pm\varepsilon) + \cdots \sim \varepsilon^{-1} w'(\pm 1) + w_1'(\pm 1) + \varepsilon w_2'(\pm 1) + \cdots\,.$$

We can now expand $v(\pm\varepsilon)$, $v'(\pm\varepsilon)$ into the formal Taylor series about $x = \pm 0$. Then

$$(3.11) \qquad v(-0) = w(-1), \qquad v(+0) = w(+1),$$

$$(3.12) \qquad w'(-1) = 0, \qquad w'(1) = 0,$$

$$(3.13) \qquad v'(-0) = w_1'(-1), \qquad v'(+0) = w_1'(1),$$

$$(3.14) \qquad v_1(-0) - v'(-0) = w_1(-1), \qquad v_1(+0) + v'(+0) = w_1(1),$$

$$(3.15) \qquad v_1'(-0) - v''(-0) = w_2'(-1), \qquad v_1'(+0) + v''(+0) = w_2'(1).$$

It follows that v satisfies equation (3.6) on each half-axis and w is a solution to the problem

$$(3.16) \qquad -w'' + \alpha \Psi w = 0, \quad \xi \in (-1, 1), \qquad w'(-1) = 0,\ w'(1) = 0.$$

Moreover both these functions satisfy coupling conditions (3.11). Problem (3.16) is decisive in our next considerations, because it contains information about a kind of the singular perturbation, namely the profile Ψ and the coupling constant α. The first and primary question is whether there exists its nontrivial solution.

3.1. **Resonant Set and Coupling Function.** Problem (3.16) can be viewed as a spectral problem with spectral parameter α. Since function Ψ is sign-changing, it is natural to introduce a space with an indefinite metric. Let $\mathcal{K}$ be the weight L_2-space with the scalar product $(f, g) = \int\limits_{-1}^{1} |\Psi|\, f\overline{g}\, d\xi$.

Let us introduce in $\mathcal{K}$ the indefinite metric $[f, g] = \int\limits_{-1}^{1} \Psi f\overline{g}\, d\xi$. Then the pair $(\mathcal{K}, [\cdot, \cdot])$ is called a *Krein space* (see [4] for the original definition). In the Krein space there exists the *fundamental symmetry* $Jf = \operatorname{sgn} \Psi \cdot f$ such that $[f, g] = (Jf, g)$ for all $f, g \in \mathcal{K}$.

Let T be a closed densely defined operator in $\mathcal{K}$. The *J-adjoint* operator of T is defined by the relation $[Tx, y] = [x, T^{[*]}y]$ for $x \in \mathcal{D}(T)$ on the set of all $y \in \mathcal{K}$ such that the mapping $x \mapsto [Tx, y]$ is a continuous linear functional on $\mathcal{D}(T)$. The operator T is called *J-selfadjoint* if $T = T^{[*]}$. The operator T is called *J-nonnegative* if $[Tx, x] \geq 0$ for $x \in \mathcal{D}(T)$.

Lemma 3.1. *Let T be a J-selfadjoint and J-nonnegative operator with a nonempty resolvent set. Spectrum $\sigma(T)$ is real, moreover, the residual spectrum is empty. If $\lambda \neq 0$ is an eigenvalue of T, then it is simple (e.i., $\ker(T - \lambda) = \ker(T - \lambda)^2$). If 0 is an eigenvalue of T, then its Riesz index is equal or less than 2, e.i., $\ker T^2 = \ker T^3$. Generally, 0 is not a simple eigenvalue* [4, p. 138].

We introduce in $\mathcal{K}$ the operator $\mathcal{T}_\Psi = -\frac{1}{\Psi(\xi)}\frac{d^2}{d\xi^2}$ with the domain

$$\mathcal{D}(\mathcal{T}_\Psi) = \{f \in \mathcal{K} \mid f \in W_2^2(-1,1), \ \Psi^{-1}f'' \in \mathcal{K}, \ f'(-1) = 0, \ f'(1) = 0\}.$$

Spectral equation $\mathcal{T}_\Psi w = -\alpha w$ is associated with (3.16). Differential equations with the indefinite weights have been widely studied [14, 32, 33, 49]. For the sake of completeness we describe the main properties of $\mathcal{T}_\Psi$.

Theorem 3.1. *For each $\Psi \in \mathcal{P}$ operator $\mathcal{T}_\Psi$ is J-selfadjoint and J-nonnegative.*

Proof. Given $f \in \mathcal{D}(\mathcal{T}_\Psi)$, we have

$$[\mathcal{T}_\Psi f, g] = -\int_{-1}^{1} f''\overline{g}\,d\xi = f(1)\overline{g'(1)} - f(-1)\overline{g'(-1)} - \int_{-1}^{1} f\,\overline{g''}\,d\xi.$$

Then identity $[\mathcal{T}_\Psi f, g] = [f, \mathcal{T}_\Psi^{[*]}g]$ holds if $g \in \mathcal{D}(\mathcal{T}_\Psi)$ and $\mathcal{T}_\Psi^{[*]}g = -\Psi^{-1}g''$. Hence $\mathcal{T}_\Psi$ is J-selfadjoint. Next, for all $f \in \mathcal{D}(\mathcal{T}_\Psi)$

$$[\mathcal{T}_\Psi f, f] = -\int_{-1}^{1} f''\overline{f}\,d\xi = \int_{-1}^{1} |f'|^2\,d\xi \geq 0,$$

so that $\mathcal{T}_\Psi$ is a J-nonnegative operator. $\qquad\qquad\square$

Theorem 3.2.   (i) *Given $\Psi \in \mathcal{P}$, the spectrum of $\mathcal{T}_\Psi$ is real and discrete. All nonzero eigenvalues are simple. If $m_0(\Psi) \neq 0$, then 0 is a simple eigenvalue. In the general case, $\ker \mathcal{T}_\Psi \neq \ker \mathcal{T}_\Psi^2$.*
(ii) *If $\Psi \in \mathcal{P}_0$, then the spectrum of $\mathcal{T}_\Psi$ has two accumulation points $-\infty$ and $+\infty$. Moreover, $\ker \mathcal{T}_\Psi \neq \ker \mathcal{T}_\Psi^2$.*

Proof. To start with, show that the resolvent set of $\mathcal{T}_\Psi$ is nonempty. Each solution of the homogenous problem

$$(3.17) \qquad g'' + i\Psi g = 0, \quad \xi \in (-1,1), \qquad g'(-1) = 0, \ g'(1) = 0,$$

satisfies equality

$$\int_{-1}^{1} |g'|^2\,d\xi - i\int_{-1}^{1} \Psi\,|g|^2\,d\xi = 0.$$

Since function Ψ is real-valued, we deduce that g is a constant function. Obviously, the zero function can only be a solution of (3.17). Then the

nonhomogeneous problem $g'' + i\Psi g = f$, $g'(-1) = 0$, $g'(1) = 0$ admits a unique solution for each $f \in \mathcal{K}$. Hence $i \in \rho(\mathcal{T}_\Psi)$ and $g = R(i, \mathcal{T}_\Psi)f$, where $R(\lambda, \mathcal{T}_\Psi)$ is the resolvent of $\mathcal{T}_\Psi$.

Therefore spectrum $\sigma(\mathcal{T}_\Psi)$ is real by Lemma 3.1 and Theorem 3.1. The compactness of $R(i, \mathcal{T}_\Psi)$ follows from the sequence of embeddings

$$\mathcal{D}(\mathcal{T}_\Psi) \subset W_2^2(-1, 1) \hookrightarrow L_2(-1, 1) \subset \mathcal{K}.$$

As a consequence we have $\sigma(\mathcal{T}_\Psi) = \sigma_p(\mathcal{T}_\Psi)$.

Each eigenvalue of $\mathcal{T}_\Psi$ is simple. Indeed, if there exist two linear independent eigenfunctions φ and ψ corresponding to eigenvalue α, then function $w(\xi) = \psi(1)\varphi(\xi) - \varphi(1)\psi(\xi)$ is nonzero since both values $\varphi(1)$, $\psi(1)$ are different from 0. But this is impossible, because $w = 0$ as a unique solution to the Cauchy problem $w'' + \alpha\Psi w = 0$, $w(1) = 0$, $w'(1) = 0$.

Assume now $\Psi \in \mathcal{P}_0$. Since potential Ψ changes a sign, the spectrum $\sigma(\mathcal{T}_\Psi)$ is unbounded in both directions [14] (see Remark 3.1). Moreover invariant subspace $\ker \mathcal{T}_\Psi^2$ is generated by eigenvector $w = 1$ and root vector

$$w_*(\xi) = \int\limits_{-1}^{\xi} (t - \xi)\Psi(t)\, dt,$$

which is a solution to the problem $w_*'' = -\Psi(\xi)$, $w_*'(-1) = 0$, $w_*'(1) = 0$. This problem admits a solution if and only if $m_0(\Psi) = 0$. In view of Lemma 3.1 there are no linear independent root vectors other than w_*. $\square$

Remark 3.1. The spectrum of $\mathcal{T}_\Psi$ is a symmetric set with respect to the origin for each *odd* profile Ψ: if α is an eigenvalue with eigenfunction $w(\xi)$, then $-\alpha$ is also an eigenvalue with eigenfunction $w(-\xi)$.

We introduce the set $\Sigma_\Psi = \{\alpha \in \mathbb{R} \colon -\alpha \in \sigma(\mathcal{T}_\Psi)\}$. We call Σ_Ψ the *resonant set* of profile Ψ. Problem (3.16) has a nontrivial solutions if and only if the coupling constant α belongs to Σ_Ψ. Suppose W_α is an eigenfunction of (3.16) with eigenvalue $\alpha \in \Sigma_\Psi$. Thus values $W_\alpha(-1)$ and $W_\alpha(1)$ are nonzero and so the quotient

$$\theta_\Psi(\alpha) = \frac{W_\alpha(1)}{W_\alpha(-1)}$$

is correctly defined. Quantity $\theta_\Psi(\alpha)$ does not depend on the choice of an eigenfunction and can be considered as function $\theta_\Psi \colon \Sigma_\Psi \to \mathbb{R}$ defined at points of the resonant set. We call θ_Ψ the *coupling function* of Ψ. Hence each profile Ψ generates the resonant set and the coupling function.

3.2. **Limit Operator** $\mathcal{H}(\alpha, \Psi)$**.** We now continue to construct the asymptotics. We distinguish two different cases. First assume α *does not belong to resonant set* Σ_Ψ. Then problem (3.16) has a trivial solution $w = 0$ only,

and so from coupling conditions (3.11) we deduce $v(-0) = v(+0) = 0$. Recalling (3.6), we obtain

$$(3.18) \qquad -v'' + Uv = \lambda v, \quad x \in \mathbb{R} \setminus \{0\}, \qquad v(0) = 0, \quad v \in L_2(\mathbb{R}).$$

Function v is nonzero by our assumption, thus λ is an eigenvalue of the problem (3.18). Therefore we can introduce limit operator $\mathcal{H}(\alpha, \Psi)$, which is the direct sum $S_- \oplus S_+$ of the Schrödinger operators acting on half-axes. Operators S_- and S_+ are associated with problems

$$(3.19) \quad \begin{cases} -v'' + Uv = \lambda v, \quad x \in \mathbb{R}_-, \\ v(0) = 0, \quad v \in L_2(\mathbb{R}_-), \end{cases} \qquad \begin{cases} -v'' + Uv = \lambda v, \quad x \in \mathbb{R}_+, \\ v(0) = 0, \quad v \in L_2(\mathbb{R}_+) \end{cases}$$

respectively, where $\mathbb{R}_-$ and $\mathbb{R}_+$ are the negative and positive half-axes. Spectra of S_- and S_+ are real, discrete and simple. Obviously,

$$\sigma(\mathcal{H}(\alpha, \Psi)) = \sigma(S_-) \cup \sigma(S_+).$$

Let us now suppose that α *belongs to resonant set* Σ_Ψ. We set $w = aW_\alpha$, where W_α is the corresponding eigenfunction of (3.16), and $a \neq 0$. We conclude from (3.11) that $v(-0) = aW_\alpha(-1)$, $v(+0) = aW_\alpha(1)$, hence that

$$(3.20) \qquad\qquad v(+0) = \theta_\Psi(\alpha)v(-0).$$

According to (3.9), (3.13) the next term w_1 of series (3.4) can be found by solving the problem

$$(3.21) \quad -w_1'' + \alpha \Psi w_1 = 0, \xi \in (-1, 1), \quad w_1'(-1) = v'(-0), w_1'(1) = v'(+0).$$

Because α is an eigenvalue of (3.16), the problem admits a solution if and only if $W_\alpha(1)v'(+0) = W_\alpha(-1)v'(-0)$. This solvability condition can be derived multiplying the equation by W_α and integrating twice by parts. It may be written in the form

$$(3.22) \qquad\qquad \theta_\Psi(\alpha)v'(+0) = v'(-0).$$

Combining (3.6), (3.20) and (3.22) we deduce that v must be an eigenfunction of the problem

$$(3.23) \qquad \begin{cases} -v'' + Uv = \lambda v, \qquad x \in \mathbb{R} \setminus \{0\}, \\ v(+0) - \theta_\Psi(\alpha)v(-0) = 0, \quad \theta_\Psi(\alpha)v'(+0) - v'(-0) = 0. \end{cases}$$

Thus the connected selfadjoint extension of operator L

$$(3.24) \quad \begin{aligned} &\mathcal{H}(\alpha, \Psi) = -\frac{d^2}{dx^2} + U(x), \\ &\mathcal{D}(\mathcal{H}(\alpha, \Psi)) = \{f \in \mathcal{D}(L^*) \colon f(+0) \\ &\qquad\qquad\qquad = \theta_\Psi(\alpha)f(-0), \, \theta_\Psi(\alpha)f'(+0) = f'(-0)\} \end{aligned}$$

is associated with (3.23).

Remark 3.2. In the case $\alpha \in \Sigma_\Psi$ we can also assume that $a = 0$, that is, $w = 0$. Then $v(0) = 0$, and so v is an eigenfunction of $S_- \oplus S_+$. But condition (3.22) has nevertheless to hold true; for otherwise we could not solve (3.21). If v is an eigenfunction corresponding to a simple eigenvalue of $S_- \oplus S_+$, then at least one of values $v'(-0)$ and $v'(+0)$ equals 0, since v vanishes on at least one of half-axes. Then condition (3.22) yields $v'(-0) = v'(+0) = 0$, which is impossible. For a double eigenvalue of $S_- \oplus S_+$, there exists a unique line in the 2-dimensional eigenspace, for which condition (3.22) can be satisfied. Vectors of this line are also eigenfunctions of (3.23), since condition (3.20) is trivially valid. Therefore in the resonant case the leading term w of (3.4) is zero if and only if $\lambda \in \sigma(\mathcal{H}(\alpha, \Psi)) \cap \sigma(S_- \oplus S_+)$. This intersection consists of all double eigenvalues of $S_- \oplus S_+$. Note that for operator $\mathcal{H}(\alpha, \Psi)$ these eigenvalues are simple.

Remark 3.3. For $\alpha = 0$ operators $\mathcal{H}_\varepsilon(0, \Psi)$ don't depend on ε and coincide with the unperturbed Schrödinger operator with potential U. On the other hand, the resonant set always contains 0, and so operator $\mathcal{H}(0, \Psi)$ must be defined as in (3.24). Obviously $\theta_\Psi(0) = 1$, because the constant eigenfunction corresponds to the zero eigenvalue of $\mathcal{T}_\Psi$. It follows that $\mathcal{H}_\varepsilon(0, \Psi) = \mathcal{H}(0, \Psi)$ for all $\varepsilon > 0$.

After all, for each profile Ψ we have defined the family of selfadjoint operators $\{\mathcal{H}(\alpha, \Psi)\}_{\alpha \in \mathbb{R}}$. If coupling constant α doesn't belong to resonant set Σ_Ψ, then operator $\mathcal{H}(\alpha, \Psi)$ is a direct sum of the Schrödinger operators acting on half-axes. The corresponding eigenfunctions describe states of a quantum system, when a particle can be found with probability 1 on one of the half-axes. In this case we obtain what will be referred to as the *closed δ'-barrier*. If $\alpha \in \Sigma_\Psi$, then operator $\mathcal{H}(\alpha, \Psi)$ given by (3.24) is the connected selfadjoint extension with matrix

$$(3.25) \qquad C_\alpha = \begin{pmatrix} \theta_\Psi(\alpha) & 0 \\ 0 & \theta_\Psi(\alpha)^{-1} \end{pmatrix}$$

in (2.2). For resonant coupling constants a particle can permeate through the barrier with some probability (the case of an *open δ'-barrier*).

4. What Is the δ'-Potential?

4.1. Historical Remarks. The formal Hamiltonians with the Dirac functions and their derivatives in potentials were studied since the eighties of last century. We refer, for example, to research monograph [1] (the first edition, 1988) and papers [50, 52, 26]. Incidentally, in solvable models one distinguishes two different phenomena dealing with the δ'-function: *the δ'-interaction* and *the point dipole interaction (the δ'-potential)*.

The first result about the δ'-interaction is due to S. Albeverio, F. Gesztesy, R. Høegh-Krohn and H. Holden [1]. The standard definition is the

one-parameter family of selfadjoint extensions

$$A_\beta = -\frac{d^2}{dx^2},$$

(4.1) $\mathcal{D}(A_\beta) = \{f \in W_2^2(\mathbb{R} \setminus \{0\})\colon f'(-0) = f'(+0),$
$$f(+0) - f(-0) = \beta f'(0)\}.$$

P. Šeba [52] showed that these selfadjoint extensions correspond to the heuristic operator

$$-\frac{d^2}{dx^2} + \beta |\delta'(x)\rangle\langle\delta'(x)|,$$

where the rank one perturbation $|f\rangle\langle f|$ is defined by

$$(|f\rangle\langle f|\varphi\rangle)(x) = f(x) \int_{\mathbb{R}} f(y)\varphi(y)\,dy.$$

In the same paper P. Šeba also studied formal Hamiltonian $-\frac{d^2}{dx^2} + \alpha\delta'(x)$, where $\delta'(x)$ was considered as the limit in the topology of $\mathcal{D}'(\mathbb{R})$ of linear combination

$$\frac{1}{2\varepsilon}\big(\delta(x+\varepsilon) - \delta(x-\varepsilon)\big).$$

The limit operator is referred to as the δ-interaction dipole Hamiltonian. The author proved that this Hamiltonian is the direct sum $S_- \oplus S_+$ with $U = 0$ (see (3.19)). For instance, in Theorem 4 one proves that the family of the Schrödinger operators with smooth potentials

(4.2) $$S_\varepsilon(V) = -\frac{d^2}{dx^2} + \frac{1}{\varepsilon^2}V\left(\frac{x}{\varepsilon}\right), \qquad x \in \mathbb{R}$$

converges to $S_- \oplus S_+$ in the norm resolvent sense, provided potential $V \in C_0^\infty(\mathbb{R})$ fulfils condition $m_0(V) = 0$. From viewpoint of the scattering theory this means that the δ' is a totally reflecting wall.

But these conclusions are at variance with the results presented in [13, 56, 58, 59], where the scattering properties of δ'-like potentials constructed in the form of squeezed rectangles were studied in the zero-range limit (a simple such problem is treated in Subsection 4.3). The resonances in the transmission probability were firstly described in [13]. The authors obtained a discrete set of resonant coupling constants α_n for which there is a partial transmission decreasing as α_n is larger. These values are roots of a transcendent equation, which depends on the regularization, that is to say, the profile of piecewise constant potentials.

It is worth to point out preprint [42]. P. Kurasov and N. Elander proposed to define the Hamiltonian with the δ'-potential as the second derivative operator in $L_2(\mathbb{R})$ subject to the coupling conditions

(4.3)
$$f(+0) - f(-0) = \frac{\alpha}{2}(f(+0) + f(-0)),$$
$$f'(+0) - f'(-0) = -\frac{\alpha}{2}(f'(+0) + f'(-0)).$$

This definition is based on the extension of Dirac's functions to the case of discontinuous test functions [43]:

$$\langle \delta^{(n)}(x), \varphi(x) \rangle = \frac{(-1)^n}{2}(\varphi^{(n)}(+0) + \varphi^{(n)}(-0)).$$

The same definition was introduced by L. Nizhnik [48], who studied the Schrödinger operator with the δ'-potential in Sobolev space $W_2^3(\mathbb{R} \setminus 0)$.

Alternative approaches to the problem can be found in [25, 30, 35]. In [3, p. 339] we read that the relation between the set of point interactions (selfadjoint extensions from Lemma 2.1) and the set of interactions defined by the δ'-potential can not be established without additional assumptions on the symmetry properties of the interaction. Different assumptions can lead to different one dimensional families of point interactions corresponding to the formal expression $-\frac{d^2}{dx^2} + \alpha\delta'(x)$. It favours the view that this problem contains hidden parameters.

4.2. Solvable Model for Hamiltonian
with Potential $\alpha\varepsilon^{-2}\Psi(\varepsilon^{-1}x)$. The formal asymptotic results obtained in the previous section point to the following fact: the solvable model that is appropriate to operator $-\frac{d^2}{dx^2} + \alpha\delta'(x)$ is not specific for the given point interaction, it depends on the way in which the derivative of Dirac's function is approximated in the weak topology. In fact, profile Ψ is a hidden parameter in the conventional formulation of problem how to define the δ'-potential.

However, the solvable model that most closely corresponds to the motion of a quantum particle in potential $\frac{\alpha}{\varepsilon^2}\Psi(\varepsilon^{-1}x)$ is uniquely defined. The model is described by the family of operators $A(\alpha, \Psi) = -\frac{d^2}{dx^2}$ with domains

$$\mathcal{D}(A(\alpha, \Psi)) = \{f \in W_2^2(\mathbb{R} \setminus 0) \colon f(-0) = f(+0) = 0\} \quad \text{for } \alpha \notin \Sigma_\Psi,$$
$$\mathcal{D}(A(\alpha, \Psi)) = \{f \in W_2^2(\mathbb{R} \setminus 0) \colon f(+0) = \theta_\Psi(\alpha)f(-0),$$
$$\theta_\Psi(\alpha)f'(+0) = f'(-0)\} \text{ for } \alpha \in \Sigma_\Psi,$$

where Σ_Ψ and θ_Ψ are the resonance set and the coupling function of Ψ.

This solvable model is in good agreement with the results obtained previously in [13, 56, 58, 59]. We actually describe the phenomenon of resonance in the transmission probability for arbitrary profiles Ψ and obtain the resonance set as a spectral characteristic of the profile.

It is worth to compare our definition with that proposed in [42, 48] (see (4.3)), in which the coupling matrix

$$C_\alpha = \begin{pmatrix} \frac{2+\alpha}{2-\alpha} & 0 \\ 0 & \frac{2-\alpha}{2+\alpha} \end{pmatrix}$$

is also diagonal provided $|\alpha| \neq 2$. In addition, for two values of α the model is described by separated selfadjoint extensions, namely $f'(-0) = 0$, $f(+0) = 0$ for $\alpha = -2$, and $f(-0) = 0$, $f'(+0) = 0$ for $\alpha = 2$.

From the physical viewpoint smooth potential U has no effect on the coupling conditions at the origin. Moreover, the resonance set and the coupling function don't depend upon U. We cannot but assume that $U = 0$ in our computations above, because operators $\mathcal{H}_\varepsilon(\alpha, \Psi)$ lose discreteness of the spectrum. We shall give two examples of exactly solvable models with piecewise constant δ'-like potentials in order to motivate our definition of the point dipole interaction.

4.3. Transition of Quantum Particle Through δ'-Like Potential.

Let us consider the equation

$$(4.4) \qquad -y'' + \alpha\varepsilon^{-2}\Psi(\varepsilon^{-1}x)y = k^2 y, \quad x \in \mathbb{R},$$

with Ψ being a δ'-like profile, such that $\Psi(\xi) = 1$ for $\xi \in (-1, 0)$ and $\Psi(\xi) = -1$ for $\xi \in (0, 1)$, $k > 0$. Clearly the results obtained above are still true for the piecewise constant potentials. Assume that $\alpha = \varkappa^2 > 0$. It is desired to find solution $y_\varepsilon(x; \varkappa, k)$ of (4.4), such that $y_\varepsilon(x; \varkappa, k) = e^{ikx} + R_\varepsilon(\varkappa, k)e^{-ikx}$ for $x < -\varepsilon$ and $y_\varepsilon(x; \varkappa, k) = T_\varepsilon(\varkappa, k)e^{ikx}$ for $x > \varepsilon$. Quantities $|R_\varepsilon(\kappa, k)|^2$, $|T_\varepsilon(\kappa, k)|^2$ are called a *reflection coefficient* and a *transmission coefficient* respectively. They have a probabilistic meaning in view of $|R_\varepsilon(\varkappa, k)|^2 + |T_\varepsilon(\varkappa, k)|^2 = 1$. The same example is treated in [13]. We adduce this example to show the direct relationship between scattering data and the coupling function introduced above.

For the profile under consideration resonant set Σ_Ψ is symmetric with respect to the origin, and its positive part consists of the roots of transcendent equation $h(\sqrt{\alpha}) = 0$, where $h(\varkappa) = \varkappa(\tanh\varkappa - \tan\varkappa)$. The coupling function is given by

$$(4.5) \quad \theta_\Psi(\alpha) = \frac{\cosh\sqrt{\alpha}}{\cos\sqrt{\alpha}} \quad \text{if } \alpha \geq 0, \qquad \theta_\Psi(\alpha) = \frac{\cos\sqrt{-\alpha}}{\cosh\sqrt{-\alpha}} \quad \text{if } \alpha < 0.$$

Let $u_1(\xi; \varkappa, \tau)$, $u_2(\xi; \varkappa, \tau)$ be the fundamental system of solutions of

$$-u'' + (\varkappa^2\Psi(\xi) - \tau^2)u = 0, \quad \xi \in (-1, 1),$$

such that its elements are smooth functions of $\varkappa$ and τ. Such solutions can be constructed explicitly with trigonometric and hyperbolic functions.

Then we can find solution $y_\varepsilon(x; \varkappa, k)$ in the form

$$y_\varepsilon(x; \varkappa, k) = \begin{cases} e^{ikx} + R_\varepsilon \, e^{-ikx} & \text{for } x < -\varepsilon, \\ C_{\varepsilon,1} \, u_1(\frac{x}{\varepsilon}, \varkappa, \varepsilon k) + C_{\varepsilon,2} \, u_2(\frac{x}{\varepsilon}, \varkappa, \varepsilon k) & \text{for } |x| < \varepsilon, \\ T_\varepsilon \, e^{ikx} & \text{for } x > \varepsilon. \end{cases}$$

where we intend to select the coefficients to satisfy the C^1-smoothness conditions at points $x = \pm\varepsilon$. A direct computation gives

$$(4.6) \qquad T_\varepsilon(\varkappa, k) = \frac{2i\varepsilon k e^{-2i\varepsilon k}}{(2i\varepsilon k - h(\varkappa)) \cos \varkappa \cosh \varkappa + O(\varepsilon^2 k^2)}$$

as $\varepsilon k \to 0$. The asymptotic behavior of the transmission coefficient as $\varepsilon \to 0$ depends upon whether $\alpha = \kappa^2$ belongs to the resonant set. Namely

$$(4.7) \qquad |T_\varepsilon(\varkappa, k)|^2 = \frac{4\varepsilon^2 k^2}{h^2(\varkappa) \cos^2 \varkappa \cosh^2 \varkappa} \cdot \left(1 + O\left(\varepsilon k\right)\right) \quad \text{for } \varkappa^2 \notin \Sigma_\Psi,$$

$$(4.8) \qquad |T_\varepsilon(\varkappa, k)|^2 = \frac{1}{\cos^2 \varkappa \cosh^2 \varkappa} \cdot \left(1 + O(\varepsilon k)\right) \quad \text{for } \varkappa^2 \in \Sigma_\Psi.$$

Whence it follows that the transmission coefficient goes to zero outside the resonant set, and it tends to a positive limit for $\alpha \in \Sigma_\Psi$. This limit $|T(\alpha)|^2$ is independent of k and can be represented in terms of the coupling function

$$|T(\alpha)|^2 = \frac{4\theta_\Psi^2(\alpha)}{(1 + \theta_\Psi^2(\alpha))^2}, \qquad \alpha \in \Sigma_\Psi.$$

This formula is still true for negative α. According to (4.5), transmission probability $|T(\alpha)|^2$ tends to zero as $|\alpha| \to +\infty$.

Therefore operators $A(\alpha, \Psi)$, defined in the previous subsection, are the *connected* selfadjoint extensions if and only if $\alpha \in \Sigma_\Psi$.

4.4. Sturm-Liouville Operator With δ'-Like Potential.

Let (a, b) be a bounded interval, which contains the origin. We consider the eigenvalue problem

$$(4.9) \quad -y'' + \alpha\varepsilon^{-2}\Psi(\varepsilon^{-1}x)y = \lambda y, \quad x \in (a, b), \qquad y(a) = 0, \quad y(b) = 0,$$

where Ψ is the piecewise constant δ'-like potential as in the previous subsection. The fundamental system of solutions of (4.9) can be constructed explicitly with trigonometric and hyperbolic functions.

We perform the calculations for positive α only. Let us introduce the notation $\omega = \sqrt{\lambda}$, $\varkappa = \sqrt{\alpha}$, $h_1(\varkappa) = \tanh \varkappa \tan \varkappa - 1$ and $g(\varkappa) = (1 + \tanh \varkappa \tan \varkappa)(1 - \tanh \varkappa \tan \varkappa)^{-1}$. The characteristic determinant $\Delta(\varepsilon, \varkappa; \omega)$ of (4.9), whose roots are eigenfrequencies ω_ε, admits the asymptotic representation as $\varepsilon\omega \to 0$:

$$(4.10) \quad \Delta(\varepsilon, \varkappa; \omega) = h(\varkappa)\{\tan a\omega \tan b\omega + \varepsilon\omega \left(\tan b\omega - \tan a\omega\right)\}$$
$$+ \varepsilon\omega \, h_1(\varkappa)\left(\tan b\omega - g(\varkappa) \tan a\omega\right) + O(\varepsilon^2\omega^2)$$

with $h = h(\varkappa)$ being the characteristic determinant of problem (3.16) defined in 4.3.

Suppose that $\alpha \notin \Sigma_\Psi$. Then $h(\varkappa)$ is different from 0, and so bounded as $\varepsilon \to 0$ eigenfrequencies ω_ε of (4.9) tend to the roots of equation

$$\tan a\omega \tan b\omega = 0.$$

The corresponding eigenfunctions y_ε converge in $C(a,b)$ and the set of their possible limits consists of two families of functions

$$y_{k,1}(x) = \begin{cases} \sin\frac{\pi k}{a}(x-a), & x \in (a,0) \\ 0, & x \in (0,b) \end{cases},$$

$$y_{k,2}(x) = \begin{cases} 0 & x \in (a,0) \\ \sin\frac{\pi k}{b}(x-b), & x \in (0,b) \end{cases},$$

where $k \in \mathbb{N}$. It is easy to check that the limit eigenfrequencies and eigenfunctions correspond to the direct sum of the second derivative operators on intervals $(a,0)$ and $(0,b)$ subject to the Dirichlet conditions. The sum is similar to operator $A(\alpha, \Psi)$ for $\alpha \notin \Sigma_\Psi$, defined in 4.2.

Now assume $\alpha \in \Sigma_\Psi$, and so $h(\varkappa) = 0$. Remark that at the same time $h_1(\varkappa) = \tanh^2 \varkappa - 1 < 0$. In this case the eigenfrequencies ω_ε go to the roots of equation $\tan b\omega = g(\varkappa) \tan a\omega$ with

$$g(\varkappa) = \frac{1 + \tan^2 \varkappa}{1 - \tanh^2 \varkappa} = \frac{\cosh^2 \varkappa}{\cos^2 \varkappa} = \theta_\Psi^2(\alpha) \quad \text{for } \alpha \in \Sigma_\Psi \cap \mathbb{R}_+.$$

The eigenfunctions y_ε converge in $L_2(a,b)$ as well as uniformly on each of intervals $(a,0)$ and $(0,b)$ towards functions of the family

$$y_k(x) = \begin{cases} r(\omega_k) \sin \omega_k(x-a), & x \in (a,0) \\ \theta(\alpha) \sin \omega_k(x-b), & x \in (0,b) \end{cases},$$

where ω_k is a root of $\tan b\omega = \theta_\Psi^2(\alpha) \tan a\omega$, and $r(\omega) = \frac{\sin b\omega}{\sin a\omega}$ for $\sin a\omega \neq 0$, $r(\omega) = \frac{b \cos b\omega}{a \cos a\omega}$ for $\sin a\omega = 0$. We check at once that ω_k^2 and y_k are eigenvalues and eigenfunctions of the problem

$$\begin{cases} -y'' = \lambda y, & x \in (a,0) \cup (0,b), \qquad y(a) = y(b) = 0, \\ y(+0) = \theta_\Psi(\alpha)y(-0), & \theta_\Psi(\alpha)y'(+0) = y'(-0), \end{cases}$$

The operator associated with the problem is similar to operator $A(\alpha, \Psi)$ in the resonant case.

4.5. **Open Problem.** In Section 3 we have constructed the asymptotics in the case of an arbitrary profile belonging to $\mathcal{P}$. The case of δ'-like potentials $\Psi \in \mathcal{P}_0$ is characterized by a special structure of resonance set Σ_Ψ described in Theorem 3.2*(ii)* as well as a behavior of coupling function θ_Ψ. The exactly solvable models above and the computer simulation of more complicated models suggest the following

Hypothesis. *Suppose that potential Ψ fulfils conditions $m_0(\Psi) = 0$, $m_1(\Psi) = -1$. Then the coupling function θ_Ψ possesses the following property:*

    ⋄ *$|\theta_\Psi(\alpha)| > 1$ for $\alpha \in \Sigma_\Psi \cap \mathbb{R}_+$ and $|\theta_\Psi(\alpha)| \to +\infty$ as $\alpha \to +\infty$,*

    ⋄ *$|\theta_\Psi(\alpha)| < 1$ for $\alpha \in \Sigma_\Psi \cap \mathbb{R}_-$ and $|\theta_\Psi(\alpha)| \to 0$ as $\alpha \to -\infty$.*

The property has its origins in the structure of eigenfunctions of J-selfadjoint and J-nonnegative operator $\mathcal{T}_\Psi$. Because of this, the demonstration could be obtained using the Krein space theory. Note that our hypothesis for the behavior of θ_Ψ fails for other classes of potentials. There exist even functions Ψ such that $m_0(\Psi) \neq 0$ as well as $m_0(\Psi) = m_1(\Psi) = 0$. But for an even potential we have $|\theta_\Psi(\alpha)| = 1$ for all $\alpha \in \Sigma_\Psi$.

The physical interpretation of a coupling function is fairly simple. Suppose that v is a normalized in $L_2(\mathbb{R})$ eigenfunction of (3.23) and $P_v(a, b) = \int_a^b |v(x)|^2 \, dx$ is the probability of finding a particle in interval (a, b) provided that the system is in a pure state v. Then

$$\theta_\Psi^2(\alpha) = \lim_{r \to +0} \frac{P_v(0, r)}{P_v(-r, 0)},$$

which is to say, $\theta_\Psi^2(\alpha)$ is the marginal ratio of probabilities of finding a particle in $(0, r)$ and $(-r, 0)$ respectively. Let us consider potential $\frac{\alpha}{\varepsilon^2}\Psi(\varepsilon^{-1}x)$ with the profile being an odd function as shown in Pic. 1 from the left. The hypothesis states that the probability to find an electron nearby the origin is higher from the side of the deep well than that of the high wall. In other words, for positive α a negative drop in the electron density occurs, while passing across the junction from the barrier to the well side. If α is negative, then positions of the barrier and the well are changed over.

5. Asymptotics of Bounded Spectrum of $\mathcal{H}_\varepsilon(\alpha, \Psi)$: Correctors

We need to find next terms of series (3.2)–(3.4) in order to justify the proximity of energy levels of Hamiltonians $\mathcal{H}_\varepsilon(\alpha, \Psi)$ and $\mathcal{H}(\alpha, \Psi)$.

5.1. **Case of Closed δ'-Barrier.** If coupling constant α doesn't belong to Σ_Ψ, then $w = 0$. Let λ be a simple eigenvalue of $\mathcal{H}(\alpha, \Psi) = S_- \oplus S_+$ with $L_2(\mathbb{R})$-normalized eigenfunction v. Without loss of generality we can assume that $\lambda \in \sigma(S_+)$. Clearly then v vanishes in $\mathbb{R}_-$. Recalling (3.9) and (3.13) one obtains

$$-w_1'' + \alpha\Psi w_1 = 0, \quad \xi \in (-1, 1), \qquad w_1'(-1) = 0, \quad w_1'(1) = v'(+0).$$

The problem admits a unique solution, since α is not an eigenvalue of (3.16). In view of (3.7), (3.14) function v_1 is a solution of the problems

$$(5.1) \quad -v_1'' + U v_1 = \lambda v_1, \quad x \in \mathbb{R}_-, \qquad v_1(-0) = w_1(-1),$$

$$(5.2) \quad -v_1'' + U v_1 = \lambda v_1 + \lambda_1 v, \quad x \in \mathbb{R}_+, \quad v_1(+0) = w_1(1) - v'(+0)$$

on each half-axis. The first of problems admits a unique solution in $L_2(\mathbb{R}_-)$ since $\lambda \notin \sigma(S_-)$. The solution of the second one does not generally exist.

According to Fredholm's alternative problem (5.2) has a solution in $L_2(\mathbb{R}_+)$ if and only if $\lambda_1 = v'(+0)(v'(+0)-w_1(1))$. To obtain this, we multiply equation (5.2) by eigenfunction v and then integrate twice by parts. Solution v_1 is ambiguously determined, namely it is defined up to term cv. Subordinating it to the condition $\int_{\mathbb{R}_-} vv_1\, dx = 0$ we fix it uniquely. Recalling (3.10), (3.15) and $w = 0$, we obtain the problem

$$-w_2'' + \alpha\Psi w_2 = 0, \quad \xi \in (-1,1),$$

(5.3)

$$w_2'(-1) = v_1'(-0), \quad w_2'(1) = v_1'(+0) + v''(+0),$$

which gives us the corrector w_2.

We introduce the notation

(5.4) $\qquad \Lambda_\varepsilon = \lambda + \varepsilon\lambda_1, \quad Y_\varepsilon(x) = \begin{cases} v(x) + \varepsilon v_1(x), & |x| > \varepsilon, \\ \varepsilon w_1(\varepsilon^{-1}x) + \varepsilon^2 w_2(\varepsilon^{-1}x), & |x| < \varepsilon \end{cases}$

for the constructed approximations of eigenvalues and eigenfunctions. Such approximations can also be found in the case, when either $\lambda \in \sigma(S_-)\backslash\sigma(S_+)$ or $\lambda \in \sigma(S_-)\cap\sigma(S_+)$.

5.2. **Case of Open δ'-Barrier.** We now assume that α belongs to resonant set Σ_Ψ, and $w = aW_\alpha$, where W_α is an eigenfunction of (3.16) and a is an arbitrary constant. In this case λ is an eigenvalue of (3.23) with $L_2(\mathbb{R})$-normalized eigenfunction v. Note that all eigenvalues of this problem are simple. The first condition (3.11) yields $a = \frac{v(-0)}{W_\alpha(-1)}$. Then the second one also holds since $v(+0) = \theta_\Psi(\alpha)v(-0) = \frac{v(-0)}{W_\alpha(-1)}W_\alpha(1) = aW_\alpha(1)$. In view of Remark 3.2 constant a is zero if and only if $\lambda \in \sigma(S_-)\cap\sigma(S_+)$.

Problem (3.21) admits solution w_1, because (3.22) holds. This solution can be represented as $w_1 = w_1^* + a_1W_\alpha$, where w_1^* is a partial solution of the problem and a_1 is an arbitrary constant. This constant will be found below, but first we shall construct v_1.

Function v_1 satisfies equation (3.7) outside of the origin and

$$v_1(+0) - \theta_\Psi(\alpha)v_1(-0) = g_1$$

with $g_1 = w_1(1) - \theta_\Psi(\alpha)w_1(-1) - v'(+0) - \theta_\Psi(\alpha)v'(-0)$, which follows from equalities (3.14). Although a_1 is as yet unknown, constant g_1 is uniquely determined. Indeed,

$$w_1(1) - \theta_\Psi(\alpha)w_1(-1) = w_1^*(1) - \theta_\Psi(\alpha)w_1^*(-1)+$$
$$+ a_1\big(W_\alpha(1) - \theta_\Psi(\alpha)W_\alpha(-1)\big) = w_1^*(1) - \theta_\Psi(\alpha)w_1^*(-1).$$

Next we employ (3.10) and (3.15) to derive the problem

(5.5) $\qquad \begin{cases} -w_2'' + \alpha\Psi(\xi)w_2 = (\lambda - U(0))w, & \xi \in (-1,1), \\ w_2'(-1) = v_1'(-0) - v''(-0), & w_2'(1) = v_1'(+0) + v''(+0). \end{cases}$

Its solvability condition can be written as

$$(5.6) \qquad \theta_\Psi(\alpha)v_1'(+0) - v_1'(-0) = h_1$$

with $h_1 = (w(-1))^{-1}(\lambda - U(0)) \int_{-1}^{1} w^2 \, d\xi - \theta_\Psi(\alpha)v''(+0) - v''(-0)$. Therefore v_1 must be a solution of the problem

$$(5.7) \qquad \begin{cases} -v_1'' + Uv_1 = \lambda v_1 + \lambda_1 v, & x \in \mathbb{R} \setminus \{0\}, \\ v_1(+0) - \theta_\Psi(\alpha)v_1(-0) = g_1, & \theta_\Psi(\alpha)v_1'(+0) - v_1'(-0) = h_1. \end{cases}$$

Free parameter λ_1 in the right-hand side of equation (5.7) enables us to solve the problem. In light of Fredholm's alternative, (5.7) admits a solution if and only if $\lambda_1 = g_1 v'(-0) - h_1 v(-0)$. For definiteness sake, the solution is subject to additional condition $\int_{\mathbb{R}} vv_1 \, dx = 0$.

Given v_1, we can now compute constant a_1. From the first condition in (3.14) we deduce $a_1 = (W_\alpha(-1))^{-1}(v_1(-0) - v'(-0) - w_1^*(-1))$. It is easy to check directly that the second condition in (3.14) also holds.

Finally in the case of open δ'-barrier one obtains such approximations for the eigenvalue and the eigenfunction of the perturbed problem

$$\Lambda_\varepsilon = \lambda + \varepsilon\lambda_1,$$

$$(5.8) \qquad Y_\varepsilon(x) = \begin{cases} v(x) + \varepsilon v_1(x), & |x| > \varepsilon, \\ \dfrac{v(-0)}{W_\alpha(-1)} W_\alpha(\varepsilon^{-1}x) + \varepsilon w_1(\varepsilon^{-1}x) + \varepsilon^2 w_2(\varepsilon^{-1}x), & |x| < \varepsilon. \end{cases}$$

Here w_2 is an arbitrary solution of (5.5). Recall that condition (5.6) ensures its existence. The choice of a_2 in representation $w_2 = w_2^* + a_2 W_\alpha$ is of no importance since we do not look for corrector v_2.

6. Justification of Asymptotic Expansions

As shown in Theorem 2.1, for some singular potential $\alpha\Psi_\varepsilon(x)$ there is finite number $N(\alpha, \Psi) > 0$ of eigenvalues $\lambda_k^\varepsilon(\alpha, \Psi)$, converging to $-\infty$ as $\varepsilon \to 0$. Other eigenvalues remain bounded as $\varepsilon \to 0$. We shall show that these eigenvalues converge to the eigenvalues of $\mathcal{H}(\alpha, \Psi)$.

6.1. **Convergence Theorem.** Let $\{\lambda_\varepsilon\}_{\varepsilon \in \mathcal{I}}$ be a sequence of eigenvalues of $\mathcal{H}_\varepsilon(\alpha, \Psi)$ and assume that $\{y_\varepsilon\}_{\varepsilon \in \mathcal{I}}$ is a sequence of the corresponding $L_2(\mathbb{R})$-normalized eigenfunctions. Here $\mathcal{I}$ is an infinite subset of $(0, 1)$ for which 0 is an accumulation point.

Theorem 6.1. *If $\lambda_\varepsilon \to \lambda$ and $y_\varepsilon \to v$ in $L_2(\mathbb{R})$ weakly as $\mathcal{I} \ni \varepsilon \to 0$, then λ is an eigenvalue of $\mathcal{H}(\alpha, \Psi)$ with the corresponding eigenfunction v. Furthermore, y_ε converges to v in the $L_2(\mathbb{R})$-norm.*

We have divided the proof into a sequence of lemmas.

Lemma 6.1. *Under the assumptions of Theorem 6.1, for every $\gamma > 0$ sequence y_ε converges to v in $W_2^2(\mathbb{R} \setminus (-\gamma, \gamma))$ weakly as well as in the*

norm of $C^1(\mathbb{R} \setminus (-\gamma, \gamma))$. *Moreover,* v *is a solution of the Schrödinger equation*

$$(6.1) \qquad\qquad -v'' + Uv = \lambda v$$

on each of half-axes.

Proof. Let $\mathcal{M}_\gamma$ be the set of test functions $\varphi \in C_0^\infty(\mathbb{R})$ such that $\varphi(x) = 0$ for $x \in (-\gamma, \gamma)$. We conclude from (3.1) that

$$(6.2) \qquad\qquad \int_\mathbb{R} y_\varepsilon'' \varphi \, dx = \int_\mathbb{R} (U - \lambda_\varepsilon) \, y_\varepsilon \varphi \, dx$$

for all $\varphi \in \mathcal{M}_\gamma$ and $\varepsilon < \gamma$ since the support of short-range potential Ψ_ε lies in $(-\gamma, \gamma)$. The right-hand side of (6.2) has a limit as $\varepsilon \to 0$ by the assumptions, thus the left-hand side also converges for all $\varphi \in \mathcal{M}_\gamma$. We deduce then that $y_\varepsilon \to v$ in $W_2^2(\mathbb{R} \setminus (-\gamma, \gamma))$ weakly and

$$\int_\mathbb{R} v'' \varphi \, dx = \int_\mathbb{R} (U - \lambda) \, v\varphi \, dx \qquad \text{for all } \varphi \in \mathcal{M}_\gamma.$$

It follows herefrom that v is a solution of (6.1) on $\mathbb{R} \setminus (-\gamma, \gamma)$, and so on whole half-lines in view of the arbitrariness of γ. It remains to note that the imbedding theorem implies convergence of y_ε in $C^1(\mathbb{R} \setminus (-\gamma, \gamma))$. $\square$

Lemma 6.2. *Suppose as in Theorem 6.1 that* $\lambda_\varepsilon \to \lambda$ *and* $y_\varepsilon \to v$ *in* $L_2(\mathbb{R})$ *weakly as* $\mathcal{I} \ni \varepsilon \to 0$. *Then* $y_\varepsilon(\varepsilon) \to v(+0)$, $y_\varepsilon'(\varepsilon) \to v'(+0)$, $y_\varepsilon(-\varepsilon) \to v(-0)$ *and* $y_\varepsilon'(-\varepsilon) \to v'(-0)$ *as* $\mathcal{I} \ni \varepsilon \to 0$.

Proof. Let ζ be the $C^\infty(\mathbb{R} \setminus \{0\})$-function such that $\zeta(x) = 0$ for $x < 0$ and $x > 2$, and also $\zeta(x) = 1$ for $x \in (0, 1)$. We hereafter denote the characteristic function of set K by χ_K. Let us introduce sequence $\zeta_\varepsilon(x) = \chi_{(\varepsilon, \infty)}(x)\zeta(x)$. Multiplying both equations (3.1), (6.1) by ζ_ε and integrating by parts yield

$$y_\varepsilon'(\varepsilon) = -\int_1^\infty y_\varepsilon' \zeta' \, dx + \int_\varepsilon^\infty (\lambda_\varepsilon - U) y_\varepsilon \zeta \, dx,$$

$$v'(\varepsilon) = -\int_1^\infty v' \zeta' \, dx + \int_\varepsilon^\infty (\lambda - U) v\zeta \, dx.$$

Here we take into account that $\zeta_\varepsilon(\varepsilon + 0) = 1$ and $\zeta_\varepsilon'(x) = 0$ for $x \in (\varepsilon, 1)$. According to Lemma 6.1 the right-hand sides of the equalities have the same limit as $\varepsilon \to 0$, then $y_\varepsilon'(\varepsilon) \to v'(+0)$. In a similar way, using cut-function $\zeta_\varepsilon(-x)$, we can prove that $y_\varepsilon'(-\varepsilon) \to v'(-0)$.

Now let us consider the cut-function $\eta \in C^\infty(\mathbb{R} \setminus \{0\})$ such that $\eta(x) = 0$ for $x < 0$ and $x > 2$, and also $\eta(x) = x$ for $x \in (0, 1)$. We set $\eta_\varepsilon(x) =$

$\chi_{(\varepsilon,\infty)}(x)\eta(x)$. In view of equalities $\eta_\varepsilon(\varepsilon + 0) = \varepsilon$ and $\zeta_\varepsilon''(x) = 0$ for $x \in (\varepsilon, 1)$, and employing (3.1), (6.1) we derive

$$y_\varepsilon(\varepsilon) = \varepsilon y_\varepsilon'(\varepsilon) - \int_1^\infty y_\varepsilon \eta'' \, dx + \int_\varepsilon^\infty (U - \lambda_\varepsilon) y_\varepsilon \eta \, dx,$$

$$v(\varepsilon) = \varepsilon v'(\varepsilon) - \int_1^\infty v \eta'' \, dx + \int_\varepsilon^\infty (U - \lambda) v \eta \, dx.$$

Because $y_\varepsilon'(\varepsilon) \to v'(+0)$, we conclude that the right-hand sides of these equalities converge to the same limit, and so $y_\varepsilon(\varepsilon) \to v(+0)$. The same proof works for the sequence $y_\varepsilon(-\varepsilon)$. $\qquad\qquad\qquad\square$

We are now in a position to describe the behavior of eigenfunctions y_ε in a neighborhood of the origin. Let w and z be solutions of the Cauchy problems on $[-1, 1]$

(6.3) $- w'' + \alpha\Psi(\xi)w = 0, \qquad w(-1) = 1, \quad w'(-1) = 0;$

(6.4) $- z'' + \alpha\Psi(\xi)z = 0, \qquad z(-1) = 0, \quad z'(-1) = v'(-0).$

Lemma 6.3. *If $\lambda_\varepsilon \to \lambda$ and $y_\varepsilon \to v$ in $L_2(\mathbb{R})$ weakly as $\mathcal{I} \ni \varepsilon \to 0$, then*

(6.5) $\left\| \varepsilon^{-1} y_\varepsilon(\varepsilon\xi) - \varepsilon^{-1} y_\varepsilon(-\varepsilon)w(\xi) - z(\xi) \right\|_{C^1([-1,1])} \to 0.$

Proof. Set $w_\varepsilon(\xi) = \varepsilon^{-1} y_\varepsilon(\varepsilon\xi) - \varepsilon^{-1} y_\varepsilon(-\varepsilon)w(\xi) - z(\xi)$. Upon substituting $\xi = \varepsilon^{-1}x$ we see that equation (3.1) may be rewritten as

$$-\frac{d^2 y_\varepsilon}{d\xi^2} + \alpha\Psi(\xi)y_\varepsilon = \varepsilon^2(\lambda_\varepsilon - U(\varepsilon\xi))y_\varepsilon.$$

Employing (6.3) and (6.4) we deduce that w_ε is a solution of the Cauchy problem

(6.6) $\begin{cases} -w_\varepsilon'' + \alpha\Psi(\xi)w_\varepsilon = f_\varepsilon(\xi), & \xi \in [-1, 1], \\ w_\varepsilon(-1) = 0, \quad w_\varepsilon'(-1) = y_\varepsilon'(-\varepsilon) - v'(-0) \end{cases}$

with $f_\varepsilon(\xi) = \varepsilon(\lambda_\varepsilon - U(\varepsilon\xi))y_\varepsilon(\varepsilon\xi)$. For every $f_\varepsilon \in L_2(-1, 1)$ there exists a unique solution $w_\varepsilon \in W_2^2(-1, 1)$. Moreover this solution satisfies the estimate

(6.7) $\|w_\varepsilon\|_{W_2^2(-1,1)} \leq C \left(\|f_\varepsilon\|_{L_2(-1,1)} + |y_\varepsilon'(-\varepsilon) - v'(-0)| \right)$

with constant C being independent of ε. Next the inequality

$$\int_{-1}^1 y_\varepsilon^2(\varepsilon\xi) \, d\xi = \frac{1}{\varepsilon} \int_{-\varepsilon}^\varepsilon y_\varepsilon^2(x) \, dx \leq \frac{1}{\varepsilon} \|y_\varepsilon\|_{L_2(\mathbb{R})} = \frac{1}{\varepsilon}$$

implies $\varepsilon^{-1/2} \|y_\varepsilon(\varepsilon\xi)\|_{L_2(-1,1)} \leq c$. Hence $\|f_\varepsilon\|_{L_2(-1,1)} \leq c\varepsilon^{1/2}$, because λ_ε is bounded. Therefore in light of Lemma 6.2 the right-hand side of (6.7)

tends to zero. The proof is complete by using the imbedding theorem $W_2^2(-1,1) \subset C^1([-1,1])$. □

Lemma 6.4. *If $\lambda_\varepsilon \to \lambda$ and $y_\varepsilon \to v$ in $L_2(\mathbb{R})$ weakly as $\mathcal{I} \ni \varepsilon \to 0$, then $y_\varepsilon \to v$ in $L_2(\mathbb{R})$.*

Proof. First we show that y_ε is bounded on $\mathbb{R}$ uniformly with respect to ε. Applying (6.5) and Lemma 6.2, we see at once that sequence y_ε is uniformly bounded on $[-\varepsilon, \varepsilon]$. Indeed

$$(6.8) \qquad \max_{x \in [-\varepsilon,\varepsilon]} |y_\varepsilon(x)| \leq c\varepsilon + |z(\varepsilon^{-1}x)|\varepsilon + |y_\varepsilon(-\varepsilon)| \, |w(\varepsilon^{-1}x)| \leq c_1.$$

Let $\Omega_\varepsilon = \mathbb{R} \setminus (-\varepsilon, \varepsilon)$. Multiplying the equation (3.1) by $\chi_{\Omega_\varepsilon}(x)y_\varepsilon(x)$ and integrating by parts yield

$$\int_{\Omega_\varepsilon} y_\varepsilon'^2 \, dx = \lambda_\varepsilon \int_{\Omega_\varepsilon} y_\varepsilon^2 \, dx - \int_{\Omega_\varepsilon} U y_\varepsilon^2 \, dx + y_\varepsilon'(-\varepsilon)y_\varepsilon(-\varepsilon) - y_\varepsilon'(\varepsilon)y_\varepsilon(\varepsilon),$$

All terms in the right-hand side are uniformly bounded with respect to ε. In particular, boundedness of the term including potential U follows from the WKB-asymptotics of y_ε for large x [29, p.55]. Hence sequence y_ε is bounded in $W_2^1(\Omega_\varepsilon)$, and so in $C(\Omega_\varepsilon)$. This and (6.8) yield $\max_{x \in \mathbb{R}} |y_\varepsilon(x)| \leq c$ with constant c being independent of ε.

Fix $\gamma > 0$. We see from Lemma 6.1 that for small ε the $L_2(\Omega_\gamma)$-norm of difference $y_\varepsilon - v$ is smaller than γ. Thus

$$\|y_\varepsilon - v\|_{L_2(\mathbb{R})} \leq \|y_\varepsilon - v\|_{L_2(\Omega_\gamma)} + \|y_\varepsilon - v\|_{L_2(-\gamma,\gamma)}$$
$$\leq (1 + 2\max_{x \in \mathbb{R}} |y_\varepsilon(x) - v(x)|)\gamma \leq C\gamma$$

with constant C being independent of ε. The proof is complete by recalling that γ can be made as small as is wished. □

Proof of Theorem 6.1. We conclude from Lemmas 6.1, 6.4 that v is a solution of the equation

$$-v'' + Uv = \lambda v, \qquad x \in \mathbb{R} \setminus \{0\}$$

and $\|v\|_{L_2(\mathbb{R})} = 1$. We are left with the task of showing that v satisfies appropriate coupling conditions at the origin. In light of lemma 6.3, we thus deduce

$$(6.9) \quad \varepsilon^{-1}\big(y_\varepsilon(\varepsilon) - y_\varepsilon(-\varepsilon)w(1)\big) \to z(1), \quad y_\varepsilon'(\varepsilon) - \varepsilon^{-1}y_\varepsilon(-\varepsilon)w'(1) \to z'(1)$$

as $\varepsilon \to 0$. According to Lemma 6.2, we have $y_\varepsilon(\varepsilon) - y_\varepsilon(-\varepsilon)w(1) \to 0$, $y_\varepsilon(-\varepsilon)w'(1) \to 0$. Therefore

$$(6.10) \qquad\qquad v(+0) - v(-0)w(1) = 0,$$

$$(6.11) \qquad\qquad v(-0)w'(1) = 0.$$

First let assume that $v(-0) \neq 0$. Then $w'(1) = 0$. By (6.3) w is a nontrivial solution of the boundary value problem $-w'' + \alpha \Psi w = 0$, $w'(-1) = 0$, $w'(1) = 0$. Hence w is an eigenfunction of operator $\mathcal{T}_\Psi$ and coupling constant α belongs to resonant set Σ_Ψ. Furthermore $\theta_\Psi(\alpha) = w(1)$ since $w(-1) = 1$ by construction. Thus (6.10) coincides with coupling condition (3.20). We obtain from the second formula in (6.9) that $v'(+0) = z'(1)$, and so z is a solution of the boundary value problem

$$(6.12) \qquad -z'' + \alpha \Psi z = 0, \qquad z'(-1) = v'(-0), \quad z'(1) = v'(+0),$$

which coincides with (3.21). Then solvability condition (3.22) must hold, which is the second coupling condition for v. Thus v is an eigenfunction of $\mathcal{H}(\alpha, \Psi)$ with the eigenvalue λ. Here λ is a limit of sequence λ_ε.

Now assume $v(-0) = 0$. Employing (6.10) yields $v(+0) = 0$. If $\alpha \notin \Sigma_\Psi$, then the proof is complete. In the case, when α belongs to Σ_Ψ, function v satisfies condition (3.20) and w is an eigenfunction of $\mathcal{T}_\Psi$. Then there exists solution z of (6.12), consequently condition (3.22) holds. This case is described in Remark 3.2. $\qquad \square$

Corollary 6.1. *Suppose that eigenvalue $\lambda_k^\varepsilon(\alpha, \Psi)$ of $\mathcal{H}_\varepsilon(\alpha, \Psi)$ is bounded from below. Then $\lambda_k^\varepsilon(\alpha, \Psi)$ has a finite limit as $\mathcal{I} \ni \varepsilon \to 0$ and this limit is a point of the spectrum of $\mathcal{H}(\alpha, \Psi)$.*

Proof. Suppose, contrary to our claim, that

$$\mu_* = \varliminf_{\varepsilon \to 0} \lambda_k^\varepsilon(\alpha, \Psi) < \varlimsup_{\varepsilon \to 0} \lambda_k^\varepsilon(\alpha, \Psi) = \mu^*.$$

The constants μ_*, μ^* are finite since $\lambda_k^\varepsilon(\alpha, \Psi)$ is a bounded function. Recall that $\lambda_k^\varepsilon(\alpha, \Psi)$ is a continuous function of $\varepsilon \in (0,1)$. Then for each $\lambda \in [\mu_*, \mu^*]$ there exists a subsequence of eigenvalues λ_ε, $\varepsilon \in \mathcal{I}$, converging to λ. For instance, set $\mathcal{I}$ can be chosen as a sequence of the roots of equation $\lambda_k^\varepsilon(\alpha, \Psi) = \lambda$ with respect to ε. Sequence $\{y_\varepsilon\}_{\varepsilon \in \mathcal{I}}$ of the corresponding normalized eigenfunctions contains a weakly convergent subsequence. By the theorem λ is an eigenvalue of $\mathcal{H}(\alpha, \Psi)$. Therefore interval $[\mu_*, \mu^*]$ belongs to spectrum $\sigma(\mathcal{H}(\alpha, \Psi))$, but this is impossible unless $\mu_* = \mu^*$. $\qquad \square$

Corollary 6.2. *For each eigenvalue λ of $\mathcal{H}(\alpha, \Psi)$ with multiplicity s there exist exactly s eigenvalues $\lambda_k^\varepsilon(\alpha, \Psi)$ of $\mathcal{H}_\varepsilon(\alpha, \Psi)$ converging to λ as $\varepsilon \to 0$.*

Proof. Operator $\mathcal{H}(\alpha, \Psi)$ has simple or double eigenvalues. Let us assume that $s = 1$, but $\lambda_k^\varepsilon(\alpha, \Psi) \to \lambda$ and $\lambda_{k+1}^\varepsilon(\alpha, \Psi) \to \lambda$ for some k. Then there exist two sequences $\{y_k^\varepsilon(x; \alpha, \Psi)\}_{\varepsilon \in \mathcal{I}}$, $\{y_{k+1}^\varepsilon(x; \alpha, \Psi)\}_{\varepsilon \in \mathcal{I}}$ of the eigenfunctions, which converge in $L_2(\mathbb{R})$ to vectors of the form $e^{i\varphi} v$. This contradicts the fact that $y_k^\varepsilon(x; \alpha, \Psi)$ and $y_{k+1}^\varepsilon(x; \alpha, \Psi)$ are orthogonal in $L_2(\mathbb{R})$ for all $\varepsilon \in \mathcal{I}$. Similar arguments apply to the case $s = 2$ (see [21] for more details). $\qquad \square$

6.2. **Quasimodes of $\mathcal{H}_\varepsilon(\alpha, \Psi)$.** We now show that each point of $\sigma(\mathcal{H}(\alpha, \Psi))$ is a limit of the eigenvalues of $\mathcal{H}_\varepsilon(\alpha, \Psi)$. Let B be a self-adjoint operator in Hilbert space H with a domain $\mathcal{D}(B)$. A pair $(\mu, u) \in \mathbb{R} \times \mathcal{D}(B)$ with $\|u\|_H = 1$ is called a *quasimode* of the operator B with an accuracy up to $\rho > 0$ if $\|Bu - \mu u\|_H \leq \rho$.

Lemma 6.5. *Suppose that the spectrum of B is discrete and simple. If (μ, u) is a quasimode of B with accuracy to $\rho > 0$, then interval $[\mu - \rho, \mu + \rho]$ contains an eigenvalue λ of B. Furthermore, if segment $[\mu - \tau, \mu + \tau]$ contains only this eigenvalue of B, then $\|u - v\|_H \leq 2\tau^{-1}\rho$, where v is an eigenfunction of B for the eigenvalue λ and $\|v\|_H = 1$.* [44, 57]

Inasmuch as we didn't construct the asymptotics for the case of double eigenvalue λ in the previous section, we assume that $\sigma(S_-) \cap \sigma(S_+) = \emptyset$ which ensures that the spectrum of $\mathcal{H}(\alpha, \Psi)$ is simple.

Let us construct the quasimodes of $\mathcal{H}_\varepsilon(\alpha, \Psi)$. Now fix some eigenvalue λ of $\mathcal{H}(\alpha, \Psi)$ with eigenfunction v, $\|v\|_{L_2(\mathbb{R})} = 1$. For each λ and v we have obtained the formal asymptotic approximations Λ_ε, Y_ε defined by either (5.4) or (5.8) depending upon α and Ψ. The task is to justify the asymptotics. Notice that in this subsection we don't distinguish the resonant and non-resonant cases. By construction Λ_ε and Y_ε satisfy equalities

$$-Y_\varepsilon'' + (U(x) - \Lambda_\varepsilon)Y_\varepsilon = \varepsilon^2 R_1(\varepsilon, x) \quad \text{for } |x| > \varepsilon,$$

$$(6.13) \quad -Y_\varepsilon'' + (U(x) + \alpha\varepsilon^{-2}\Psi(\varepsilon^{-1}x) - \Lambda_\varepsilon)Y_\varepsilon = \varepsilon R_2(\varepsilon, x) \quad \text{for } |x| < \varepsilon,$$

$$[Y_\varepsilon]_{x=\pm\varepsilon} = \varepsilon^2 r_1^\pm(\varepsilon), \qquad [Y_\varepsilon']_{x=\pm\varepsilon} = \varepsilon^2 r_2^\pm(\varepsilon)$$

with functions R_j, $r_j^\pm$ being uniformly bounded with respect to its arguments. Function Y_ε doesn't belong to the domain of $\mathcal{H}_\varepsilon(\alpha, \Psi)$, because it has jump discontinuities at points $x = \pm\varepsilon$. But we can construct function ζ_ε with the following properties

- ζ_ε is a smooth function out of points $x = \pm\varepsilon$ and differs from zero only for $\varepsilon < |x| < 1$;
- $[\zeta_\varepsilon]_{x=\pm\varepsilon} = -r_1^\pm(\varepsilon)$ and $[\zeta_\varepsilon']_{x=\pm\varepsilon} = -r_2^\pm(\varepsilon)$;
- $\displaystyle\max_{\varepsilon<|x|<1}\left(|\zeta_\varepsilon(x)| + |\zeta_\varepsilon'(x)| + |\zeta_\varepsilon''(x)|\right) \leq c$ with constant c being independent of ε,

which eliminates this defect. In fact, function $Y_\varepsilon + \varepsilon^2\zeta_\varepsilon$ is continuously differentiable at $x = \pm\varepsilon$ and belongs to $\mathcal{D}(\mathcal{H}_\varepsilon(\alpha, \Psi))$. We set $\Upsilon_\varepsilon = \|Y_\varepsilon + \varepsilon^2\zeta_\varepsilon\|_{L_2(\mathbb{R})}^{-1}(Y_\varepsilon + \varepsilon^2\zeta_\varepsilon)$ and substitute Υ_ε into (6.13) instead of Y_ε. Then the orders of smallness of right-hand sides in (6.13) don't change since $\|Y_\varepsilon + \varepsilon^2\zeta_\varepsilon\|_{L_2(\mathbb{R})} \to 1$ as $\varepsilon \to 0$. Therefore pair $(\Lambda_\varepsilon, \Upsilon_\varepsilon)$ is a quasimode of $\mathcal{H}_\varepsilon(\alpha, \Psi)$ with accuracy to ε.

Lemma 6.6. *For each $\lambda \in \sigma(\mathcal{H}(\alpha, \Psi))$ there is an eigenvalue $\lambda_j^\varepsilon(\alpha, \Psi)$ of $\mathcal{H}_\varepsilon(\alpha, \Psi)$ such that $\lambda_j^\varepsilon(\alpha, \Psi) \to \lambda$. Moreover,*

$$(6.14) \qquad \left| \lambda_j^\varepsilon(\alpha, \Psi) - \lambda \right| \leq c_1 \varepsilon, \qquad \left\| y_j^\varepsilon(\,\cdot\,; \alpha, \Psi) - v \right\|_{L_2(\mathbb{R})} \leq c_2 \varepsilon,$$

where y_j^ε and v are the corresponding normalized eigenfunctions.

Proof. Let $(\Lambda_\varepsilon, \Upsilon_\varepsilon)$ be the quasimode of $\mathcal{H}_\varepsilon(\alpha, \Psi)$ corresponding to the limit eigenvalue λ and eigenfunction v. According to Lemma 6.5 there exists number j such that $\left| \lambda_j^\varepsilon(\alpha, \Psi) - \Lambda_\varepsilon \right| \leq c_1 \varepsilon$, from which the first inequality in (6.14) follows. In view of Corollaries 6.1, 6.2 index j is independent of ε. If τ is less than the distance from λ to the rest of the spectrum of $\mathcal{H}(\alpha, \Psi)$, then for small enough ε interval $[\lambda - \tau, \lambda + \tau]$ contains eigenvalue $\lambda_j^\varepsilon(\alpha, \Psi)$ only. Applying again Lemma 6.5 yields $\left\| y_j^\varepsilon(\,\cdot\,; \alpha, \Psi) - \Upsilon_\varepsilon \right\|_{L_2(\mathbb{R})} \leq 2\tau^{-1} c_1 \varepsilon$, from which the second inequality in (6.14) immediately follows. $\square$

We are now in a position to state the main result. Let $\{\lambda_k(\alpha, \Psi)\}_{k=1}^\infty$ be the eigenvalues of $\mathcal{H}(\alpha, \Psi)$ numbered by increasing and let $\{v_k(x; \alpha, \Psi)\}_{k=1}^\infty$ be the orthonormal system of eigenfunctions. Recall that $N = N(\alpha, \Psi)$ is the number of eigenvalues of $\mathcal{H}_\varepsilon(\alpha, \Psi)$ converging to $-\infty$ as $\varepsilon \to 0$.

Theorem 6.2. *Assume $(\alpha, \Psi) \in \mathbb{R} \times \mathcal{P}$ and $\sigma(S_-) \cap \sigma(S_+) = \emptyset$. Then for every natural k we have*

$$(6.15) \qquad \left| \lambda_{k+N}^\varepsilon(\alpha, \Psi) - \lambda_k(\alpha, \Psi) \right| \leq c_1 \varepsilon,$$

$$(6.16) \qquad \left\| y_{k+N}^\varepsilon(\,\cdot\,; \alpha, \Psi) - v_k(\,\cdot\,; \alpha, \Psi) \right\|_{L_2(\mathbb{R})} \leq c_2 \varepsilon$$

with constants c_1, c_2 being independent of ε.

Proof. Theorem 6.1, Corollary 6.2 and Lemma 6.6 may be summarized by saying that eigenvalue $\lambda_{k+N}^\varepsilon(\alpha, \Psi)$ tends to $\lambda_k(\alpha, \Psi)$ as $\varepsilon \to 0$. Thus estimates (6.15), (6.16) are more precise versions of (6.14). $\square$

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